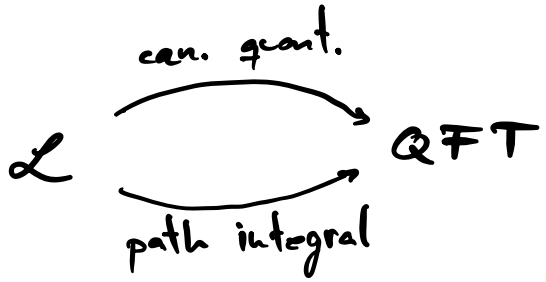
Path integral formulation of QFT (I)

- Cheng-Li : 1.2
- Srednicki : 6, 7, 8, 9
- Schwartz : 14
- P&S : 9

Our goal is to describe the dynamics of a quantum system.

You are already familiar with one formulation of QFT. One starts with the classical lagrangian $\mathcal{L}[\phi]$ of a classical field theory. Then one identifies the conjugate momenta, imposes the commutation relations and so on.

An alternative route is via the path integral formulation of QM.



Both ways to obtain a QFT from a classical field theory are thought to be equivalent.

This is hard to prove (maybe some expert knows, perhaps in lower d), mainly because the space of theories accessible via path interals is much more vast than the one accessible by canonical quantization.

For instance, theories with derivative interactions are tricky to quantize canonically, due to difficulties identifying the canonical momenta. This is the case of scalar QED (see Itzykson & Zuber, 6-1-4 for a discussion of subtelties in the canonical quant. of theories with derivative interactions)

Moreover, non-perturbative and topological phenomena, like anomalies, solitons, instantons, etc, are most naturally discussed from the path integral.

The logic we will follow here is to derive the path integral formulation from the canonical formulation, in a case where both approaches are known to lead the same results.

Then, we generalize the result to an arbitrary Lagrangian, and take this to be the definition of our QFT, without worrying about the canonical quantization of the theory.

■ PATH INTEGRAL IN QM

Consider a QM system with coordinates q_i, $i=1,\dots,N$

We assume the lagrangian to be

$$\mathcal{L} = \frac{1}{2} m \sum_i \dot{q}_i^2 - V[q_i]$$

with $\dot{q}_i = \frac{dq_i}{dt}$, V function of coord only.

The conjugate momenta and the Hamiltonian are

$$p_i = \frac{d\mathcal{L}}{d\dot{q}_i} = m\dot{q}_i$$

$$H(p_i, q_i) = \sum_i \frac{p_i^2}{2m} + V[q_i]$$

We quantize the system by promoting p & q to operators $\hat{p}, \hat{q}$ and imposing canonical commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \; ; \; [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

This system is a particle in N dimensions, or a discretized version of a real scalar field as we shall see.

- The fundamental quantity in QM is the transition matrix, corresponding to the overlap between initial and final states.

In the Schrodinger picture, where state evolve in time but operators don't, the time evolution is

$$|\psi(t)\rangle = e^{-i\frac{H(t+T)}{\hbar}} |\psi(-T)\rangle$$

Suppose that $|\psi(-T)\rangle$ is eigenstate of q,

$$|\psi(-T)\rangle = |\vec{q}^{\,in}\rangle \; ; \; \hat{q}_i |\vec{q}^{\,in}\rangle = q_i |\vec{q}^{\,in}\rangle$$

then the transition amplitude for the state to be observed in the config. $|\vec{q}^{\,out}\rangle$ is

$$K(\vec{q}^{\,out}, T; \vec{q}^{\,in}, -T) = \langle \vec{q}^{\,out} | e^{-\frac{2iT}{\hbar} H} | \vec{q}^{\,in} \rangle .$$

The path integral formalism leads to the following expression for K:

$$K(\vec{q}^{\,out}, T; \vec{q}^{\,in}, -T) = \int \mathcal{D}[\vec{q}(\cdot)]\, e^{\frac{i}{\hbar}\int_{-T}^{T} d\tau\, L(q, \dot{q})}$$

$$= \int \mathcal{D}[\vec{q}(\cdot)]\, e^{\frac{i}{\hbar} S[\vec{q}(\cdot); T, -T]}$$

The lagrangian density $L(q, \dot{q})$ is classical lagrangian, obtained from evaluating the Hamiltonian at each slice δt.
The action S depends on the endpoints, and also on the path $\vec{q}$ taken. The integral is over all paths. The "$\vec{q}(\cdot)$" means that the integral is over the functional form of the trajectory.

- The path integral measure is a complicated object. It can be defined as a limit of the trajectory in I intervals. We also have N

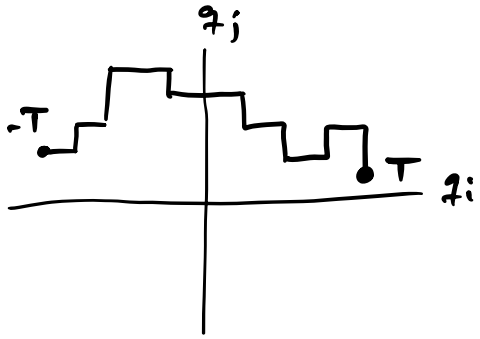
components $q_i(\cdot)$, so

$$\mathcal{D}[\vec{q}] = \prod_{i=1}^{N} \mathcal{D}[q_i].$$

with

$$\mathcal{D}[q_i] = \lim_{I\to\infty} \frac{1}{A} \prod_{K=2}^{I} \frac{dq_i^{(K)}}{A}$$

The discretized trajectory is, for instance,



$$q_i^{(K)} = q_i(-T + (K-1)\varepsilon)$$

$$; \quad \varepsilon = \frac{2T}{I}$$

The $K=1$ point is not integrated & fixed by boundary conditions.

The value of A is $A = \sqrt{\frac{2\pi i\hbar\varepsilon}{m}}$ but inconsequential. As we shall see, the normalization of the measure will not matter.

■ THE EUCLIDEAN PATH INTEGRAL

The evolution of a state with e^{iHt} implies that states receive a time-dependent phase, which is imediately unitary. However, it

leads to integrals of the type $\int e^{ix^2}$, with rapidly oscillating phases.

These are actually defined as the analytical continuation of integrals like $\int e^{-x^2}$. The same is true for the path integral, which is defined as the analytical continuation of the Euclidean path integral.

The Euclidean transition matrix discussed above is

$$K_E(T_E) = \langle out | e^{-\frac{2T_E}{\hbar} H} | in \rangle$$

$$= \int D[\vec{q}] \, e^{-\frac{1}{\hbar} S[\vec{q}; T_E]}$$

with the Euclidean action

$$S_E[\vec{q}; T_E] = \int_{-T_E}^{T_E} d\tau \left[\frac{1}{2} m \left(\frac{d\vec{q}}{d\tau} \right)^2 + V[\vec{q}] \right]$$

The path integral has to be done with boundary conditions

$$\vec{q}(-T_E) = \vec{q}^{\,in} \quad ; \quad \vec{q}(T_E) = \vec{q}^{\,out}$$

• The Euclidean action can be obtained from the Minkowskian classical action $S = \int dt\, L(q, \dot{q})$ by a replacement rule

$$\text{"}t \to -i\tau\text{"} = \begin{cases} t \to -i\tau \\ \frac{d}{dt} \to i\frac{d}{d\tau} \end{cases}$$

If we take

$$S = \int dt \left\{ \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V[q] \right\} \xrightarrow{\text{"}t\to -i\tau\text{"}} -i \int d\tau \left[-\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 - V[q] \right]$$

$$= +i\, S_E$$

For a general theory, we will define the Euclidean action in terms of this rule

$$S_E = -iS \quad \text{with "}t \to -i\tau\text{"}.$$

The goal will be to compute the Euclidean transition amplitude K_E from the Euclidean action S_E.

The transition amplitude K for $T \in \mathbb{R}^+$ is uniquely determined in terms of K_E for $T_E = +iT$.

• The Euclidean path integral is also used to compute n-point correlators. Dynamical problems can be studied through correlators of Heisenberg operators.

$$\hat{\vec{q}}_H(t) = e^{i\frac{Ht}{\hbar}} \hat{\vec{q}}(0)\, e^{-i\frac{Ht}{\hbar}}.$$

namely

$$G^{(n)}_{i_1,\dots,i_n}(t_1,\dots,t_n) \equiv \langle 0|\, T\left[\hat{q}_{i_1}(t_1)\cdots\hat{q}_{i_n}(t_n)\right]|0\rangle$$

where $|0\rangle$ is the lowest state of the theory in QM or "the vacuum" in QFT.

From the Euclidean path integral we get

$$G^{(n)}_{E;\,i_1,\dots,i_n}(t_1,\dots,t_n) \equiv \langle 0|\, T\Big[e^{\frac{t_{E,1}}{\hbar}H} \hat{q}_{i_1}(0)\, e^{-\frac{t_{E,1}-t_{E,2}}{\hbar}H} \hat{q}_{i_2}(0)\cdots$$
$$\cdots \hat{q}_{i_n}(0)\, e^{-\frac{t_{E,n}}{\hbar}H}\Big]|0\rangle$$

The time ordering in Euclidean time is defined like for real time t:

$$T\left[A(t_{E,1})B(t_{E,2})\right] = \begin{cases} A(t_{E,1})B(t_{E,2}) & \text{if } t_{E,1} > t_{E,2} \\ B(t_{E,2})A(t_{E,1}) & \text{if } t_{E,2} > t_{E,1} \end{cases}$$

Notice that the time ordering ensures that integrals are well behaved since $e^{-\Delta t H}$ terms have $\Delta t > 0$.

The correlator G is obtained from G_E via analytical continuation, since it is analytic if $t_1 > \dots > t_n$. The ordering is unchanged.

- The Euclidean correlator is obtained from the path integral as

$$G^{(n)}_{E;i_1,\dots,i_n}(t_{E,1},\dots,t_{E,n}) = \frac{\int \mathcal{D}[\vec{q}]\, q_{i_1}(t_{E,1}) \cdots q_{i_n}(t_{E,n})\, e^{-\frac{S_E[\vec{q}]}{\hbar}}}{\int \mathcal{D}[\vec{q}]\, e^{-\frac{S_E}{\hbar}}}$$

The ratio implies that overall normalization of path integral measure will turn out to be irrelevant.

We have dropped the boundary condition at $T \neq -T$. The reason is that in the above equation we took the $T \to \infty$ limit. In this limit, the result is independent of the boundary condition as long as there is a non zero overlap with the vacuum.

$$\langle 0 | \vec{q}^{\,in} \rangle \neq 0 \quad ; \quad \langle 0 | \vec{q}^{\,out} \rangle \neq 0$$

We can therefore take $\vec{q}^{\,in} = \vec{q}^{\,out} = \langle 0 | \hat{\vec{q}} | 0 \rangle$. In QFT, this is equivalent to set the field in the boundary to some solution of the classical theory.

■ PATH INTEGRAL FOR A SCALAR FIELD THEORY

Consider now a scalar field theory with no derivative interactions. We will see how it is a trivial extension of the QM case.

We will define the theory on a lattice,

$$\vec{x}_{\vec{n}} = a\,\vec{n} = a\,\{n_x, n_y, n_z\}$$

with $n_i \in [-2N, 2N]$. At each time t, the real scalar field is discretized on the lattice. The coordinates of the classical system are the values at each lattice site.

$$\varphi_{\vec{n}}(t) \equiv \varphi(x_{\vec{n}}, t)$$

This is, we have $(2N+1)^3$ coordinate

. We just have to apply the general result.

The Lagrangian is

$$L = \int d^3x \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V[\varphi] \right)$$

$$= \int d^3x \left(\frac{1}{2} (\partial_0 \varphi)^2 - \frac{1}{2} (\partial_x \varphi)^2 - \frac{1}{2} (\partial_y \varphi)^2 - \frac{1}{2} (\partial_z \varphi)^2 - V[\varphi] \right)$$

so the discretized version is given by

$$L_d = a^3 \sum_{\vec{n}} \left\{ \frac{1}{2} \dot{\varphi}_{\vec{n}}^2 - \frac{1}{2} \left(\frac{\varphi_{\vec{n}+(1,0,0)} - \varphi_{\vec{n}}}{a} \right)^2 - \frac{1}{2} (\cdots) - \frac{1}{2} (\cdots) \right.$$

$$\left. - V[\varphi] \right\}$$

Given that there are no derivative interactions, all terms except $\dot{\varphi}_{\vec{n}}^2$ are a "potential" interaction and therefore the Lagrangian is of the form assumed previously to derive the path integral formulas.

. The continuum limit is given by $a \to 0$. Doing so leads to a continuum QFT in finite volume. One needs to take $L = 4aN \to \infty$ as well.

. The Euclidean action is given by

$$S_E = \int_{-\infty}^{\infty} d\tau \sum_{\vec{n}} \left[\frac{1}{2} a^3 \dot{\varphi}_{\vec{n}}^2 + \frac{a^3}{2} \left(\frac{\varphi_{\vec{n}+(1,0,0)} - \varphi_{\vec{n}}}{a} \right)^2 + \dots + a^3 V[\varphi_{\vec{n}}] \right]$$

$$\downarrow \begin{matrix} a \to 0 \\ N \to \infty \end{matrix}$$

$$= \int d^3x \, d\tau \left[\frac{1}{2} (\partial_\tau \varphi)^2 + \frac{1}{2} (\partial_x \varphi)^2 + \dots + V[\varphi] \right]$$

It is natural to introduce Euclidean "space-time" coordinates

$$x_E^\mu = (\tau, \vec{x})$$

with the Euclidean metric $\eta_{\mu\nu}^{(E)} = \delta_{\mu\nu}$.

$$S_E = \int d^4x_E \left[\frac{1}{2} (\partial_\mu^{(E)} \varphi)(\partial^{(E)\mu} \varphi) + V[\varphi] \right]$$

. The Euclidean action can be obtained from the (true) Minkowskian action by the rule

$$\text{"}t \to -i\tau\text{"} = \begin{cases} x^0 = t \to -i\tau = -i x_E^0 \\ \frac{\partial}{\partial x^0} = \partial_0 \to i \partial_0^{(E)} \\ x^i \to x_E^i \\ x_i = -x^i \to -x_{i,E} = -x_E^i \end{cases}$$

In particular, the kinetic term is

$$(\partial\!\!\!/\varphi)(\partial\!\!\!/\varphi) \longrightarrow -\partial\!\!\!/_E^{(E)}\varphi\,\partial\!\!\!/^{(E)}\varphi$$

and

$$S_E = -iS \quad \text{with} \quad "t \to -i\tau".$$

- The formulation of a QFT using a path integral goes as follows:

  1) take <u>any</u> Lagrangian
  2) Derive the Euclidean action by "$t \to -i\tau$".
  3) Compute correlators via

$$\langle 0|T\{\varphi(x_1)\cdots\varphi(x_n)\}|0\rangle = \left.\frac{\int D\varphi\,\varphi(x_1^E)..\varphi(x_n^E)\,e^{-S_E}}{\int D\varphi\, e^{-S_E}}\right|_{\substack{\text{A.C.}\\ t_E = it}}$$

- The "$D\varphi$" symbol hides a high complexity. In QM we had

$$D[\vec{q}(\cdot)] \propto \prod_{i=1}^{N} \lim_{I\to\infty} \prod_{k=2}^{I} dq_i^{(k)}$$

but now, we have

$$D[\varphi(\cdot)] \propto \lim_{N\to\infty} \lim_{a\to 0} \lim_{I\to\infty} \prod_{\vec{n}} \prod_{K} d\varphi_{\vec{n}}^{(K)}$$

with the $T \to \infty$ limit on top.

The path integral is a formal object, we are interested on its properties and will rarely have to perform any integral!

- This does not mean that we cannot perform the integral. In fact, there is an entire field, "lattice" or "lattice QCD", whose aim is to compute the path integral above. Lattice QFT can study some questions for which perturbation theory fails, like low-energy QCD.

- The path integral & its lattice implementation cannot deal with non-renormalizable theories, like $\lambda \phi^4$, QED, Fermi theory or gravity. The discretized path integral, with $a \neq 0$, is well-defined for any reasonably (positive definite, etc) Lagrangian. However, for the theories above, the high energy limit is not well defined and the lattice does not enhance the "predictive power".

Lattice provides a useful <u>low-energy</u> description of the system, even if non-perturbative.

- The formulation allows to deal with <u>any</u> Lagrangian (pos. def etc.). For instance,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \varphi^2(\partial_\mu\varphi\partial^\mu\varphi)^4\Box\varphi$$

can be studied perturbatively without much trouble.

- We can also include vector fields A^μ. The only remark is that one should employ Euclidean vector fields A^μ_E,

$$A_\mu = \{A_0, \vec{A}\} \longrightarrow A_{E\mu} = \{A_{E0}, \vec{A}_E\}$$

and we express $S_E[A_E]$ in terms of it. The rule becomes

$$\text{"}t \to -i\tau\text{"} = \begin{cases} x^0 \to -ix^0_E \\ \partial_0 \to i\partial_0^{(E)} \\ x^i \to x^i_E \\ x_i \to -x^i_E \end{cases} \oplus \begin{cases} A_0 \to i A_{E,0} \\ A^0 \to i A^0_E \\ A_i \to A_{E,i} \\ A^i \to -A^i_E \end{cases}$$

Then, the Euclidean Maxwell action is

$$S_E = \left\{ -i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \quad \text{with "} t \to -i\tau \text{"} \right\}$$

$$= -\int d^4x_E \left(-\frac{1}{4} \left(F_{E,0i} F_E^{0i} + F_{E,i0} F_E^{i0} \right) \right.$$

$$\left. -\frac{1}{4} \left(F_{E,ij} F_E^{ij} \right) \right)$$

$$= \int d^4x_E \; \frac{1}{4} F_{E\mu\nu} F_E^{\mu\nu}$$